JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 40, 36–38 (1972)

A Matrix Decomposition Theorem

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We state and prove the following theorem relating to matrix decomposition:

THEOREM. *Any matrix M of dimension $n \times n$ can be written in a unique manner as*

$$M = \sum_{k,l=0}^{n-1} a_{kl} B^k C^l,$$

where the quantities occurring on the right hand side are defined as follows:
B is the diagonal matrix

$$\begin{bmatrix} 1 & & & & & \\ & \omega & & & & \\ & & \omega^2 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \omega^{n-1} \end{bmatrix}$$

ω being the primitive n-th root of unity. C is the primitive cyclic matrix of dimension $n \times n$

$$\begin{bmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ & 0 & 1 & \cdot & \cdot & \cdot & \cdot \\ & & 0 & 1 & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot & 0 \\ & & & & & \cdot & 1 \\ 1 & & & & & & 0 \end{bmatrix}$$

The n^2 matrices $B^k C^l$ represent all possible products of all possible powers of B and C which are roots of the unit matrix.

a_{kl} $(k = 0, 1, \ldots, n-1;\ l = 0, 1, \ldots, n-1)$ *are the elements of a matrix* A *defined uniquely by the relation*

$$R = SA, \qquad A = S^{-1}R,$$

where S *is the matrix obtained by writing the elements of* $I, B, B^2, \ldots, B^{n-1}$ *as the elements of its columns*

$$S = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-1)} \\ \vdots & & & & \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)^2} \end{bmatrix}$$

$$S^{-1} = \frac{1}{n} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{n-1} & \omega^{n-2} & \cdots & \omega \\ 1 & \omega^{2(n-1)} & \omega^{2(n-2)} & \cdots & \omega^2 \\ \vdots & & & & \\ 1 & \omega^{(n-1)^2} & \omega^{(n-1)(n-2)} & \cdots & \omega^{n-1} \end{bmatrix}$$

R *is the matrix obtained by rearranging the elements of* M *according to the following prescription:*

M *can be written uniquely as*

$$M = D_0 + D_1C + D_2C^2 + \cdots + D_{n-1}C^{n-1},$$

where D_0, D_1, D_2,... *are diagonal matrices with their elements corresponding to the elements of* M *in the positions of the nonzero elements of* $I, C, C^2, \ldots, C^{n-1}$, *respectively.*

R *is now defined as the matrix with elements of its* $(k+1)$-th *column being chosen as the elements of* D_k $(k = 0, 1, \ldots, n-1)$.

Proof. The proof of the theorem follows immediately on observing that any diagonal matrix can be expressed as a linear combination of n linearly independent diagonal matrices and without loss of generality can be expressed as

$$D = \sum_0^{n-1} a_k B^k.$$

If the elements of D are arranged as a column vector $\mathbf{r}$, then we can write

$$\mathbf{r} = S\mathbf{a},$$

where the elements of $\mathbf{a}$ are a_k $(k = 0, 1, \ldots, n-1)$ and S is the matrix with the elements of its $(k+1)$-th column being chosen as the elements of the diagonal matrix B^k.

Applying these considerations to D_0, D_1, D_2,..., D_{n-1}, we obtain

$$R = SA.$$

The matrix S has been known in matrix theory as the Sylvester matrix but here it has been obtained by a rearrangement of the diagonal matrices, a procedure which has not been used in matrix literature till now. The decomposition theorem promises to be useful since the matrices B and C have the interesting property

$$CB = \omega BC, \qquad B^n = C^n = 1.$$

Thus this theorem amounts to a "soft completion of the face" of the generalised Clifford Algebra which has been revealed to us only in recent years.